

ITÔ'S DIFFUSION IN MULTIDIMENSIONAL SCATTERING WITH SIGN-INDEFINITE POTENTIALS.

SERGEY A. DENISOV

ABSTRACT. This paper extends some results of [2] to the case of sign-indefinite potentials by applying methods developed in [3]. This enables us to prove the presence of a.c. spectrum for the generic coupling constant.

INTRODUCTION

In this paper, we consider the Schrödinger operator

$$(0.1) \quad H_\lambda = -\Delta + \lambda V, \quad x \in \mathbb{R}^3$$

where V is real-valued potential and λ is a real parameter usually called a coupling constant. We will study the dependence of the absolutely continuous spectrum of H on the behavior of potential V by blending the methods of two papers [2] and [3]. This question attracted much attention recently which resulted in many publications (see, e.g. [2]-[6], [9]-[10], [14]-[15]) due to new very fruitful ideas from approximation theory finding the way in the multidimensional scattering problems.

In [2], we introduced the stochastic differential equation that has random trajectories as its solutions. These trajectories are natural for describing the scattering properties of (0.1) provided that $\lambda V \geq 0$. The reason why this requirement was made is rooted in the method itself. Indeed, as $\lambda V \geq 0$, we have $\sigma(H_\lambda) \subseteq [0, \infty)$ for the spectrum of H_λ . Moreover, the Green's function $L(x, y, k) = (-\Delta + \lambda V - k^2)^{-1}(x, y, k)$ can be analytically continued in k to the whole upper half-plane \mathbb{C}^+ and methods of the complex function theory can be used then. If V is sign-indefinite, the negative spectrum might occur, which is hard to control, and this approach breaks down. The paper [3] (see also [15]) however develops new technique which allows to overcome this difficulty by complexifying the coupling constant and considering the hyperbolic pencil

$$P_\lambda(k) = -\Delta + k\lambda V - k^2$$

instead. Then, provided that the spacial asymptotics for the new Green's kernel $P_\lambda^{-1}(k)(x, y, k)$ is established, we can conclude that the a.c. spectrum of H_λ contains $[0, \infty)$ for a.e. λ . In the next section, we state the main result of [2] and explain how it can be generalized to the sign-indefinite case.

We are going to use the following notation. Let $\omega_R(r)$ be infinitely smooth function on \mathbb{R}^+ such that $\omega_R(r) = 1$ for $r < R - 1$, $\omega_R(r) = 0$ for $r > R + 1$,

1991 *Mathematics Subject Classification*. Primary: 35P25, Secondary: 31C15, 60J45.

Key words and phrases. Absolutely continuous spectrum, Schrödinger operator, Itô stochastic calculus, Feynman-Kac type formulae.

and $0 \leq \omega_R(r) \leq 1$. The function

$$L^0(x, y, k) = \frac{e^{ik|x-y|}}{4\pi|x-y|}$$

denotes the Green's function of the free 3d Schrödinger operator, i.e. the kernel of $R_0(k^2) = (-\Delta - k^2)^{-1}$ when $\text{Im } k > 0$. The standard symbol B_t stands for the 3-dimensional Brownian motion and \mathbb{S}^2 denotes the two-dimensional unit sphere.

We consider the three-dimensional case only as it makes the writing easy. The method however can be applied for any $d > 1$. We will often suppress the dependence on λ unless we want to emphasize it.

1. MAIN RESULT

We start with stating some results from [2]. Consider the Lipschitz vector field

$$p(x) = \left(\frac{I'_\nu(|x|)}{I_\nu(|x|)} - \nu|x|^{-1} \right) \cdot \frac{x}{|x|}, \quad \nu = 1/2$$

where I_ν denotes the modified Bessel function [1, Sect. 9.6]. Then, fix any point $x^0 \in \mathbb{R}^3$ and consider the following stochastic process

$$(1.1) \quad dX_t = p(X_t)dt + dB_t, \quad X_0 = x^0$$

with the drift given by p . The solution to this diffusion process exists and all trajectories are continuous and escape to infinity almost surely. One of the main results in [2] states (assume here that $\lambda = 1$)

Theorem 1.1 ([2]). *Let V be any continuous nonnegative function. Assume that $f \in L^2(\mathbb{R}^3)$ is nonnegative and has a compact support. Let σ_f be the spectral measure of f with respect to H_V and σ'_f be the density of its a.c. part. Then we have*

$$(1.2) \quad \exp \left[\frac{1}{2\pi} \int_{\mathbb{R}} \frac{\log \sigma'_f(k^2)}{1+k^2} dk \right] \geq C_f \int f(x^0) \mathbb{E}_{x^0} \left[\exp \left(-\frac{1}{2} \int_0^\infty V(X_\tau) d\tau \right) \right] dx^0$$

where the constant $C_f > 0$ does not depend on V .

The natural corollary of this theorem is a statement that the a.c. spectrum of H fills all of \mathbb{R}^+ provided that the potential V is summable along the trajectory X_t with positive probability (which is the same as saying that there are “sufficiently many” paths over which the potential is summable).

We are going to prove the following

Theorem 1.2. *Assume that V is bounded and continuous on \mathbb{R}^3 and*

$$(1.3) \quad \int_0^\infty |V(X_t)| dt < \infty$$

with positive probability. Then \mathbb{R}^+ supports the a.c. spectrum of H_λ for a.e. λ .

Remark. The conditions of continuity and boundedness of potential are assumed for simplicity only and can probably be relaxed. The condition of λ being generic is perhaps also redundant but this method does not yield any

result for a particular value of $\lambda \neq 0$. Under the conditions of the theorem, the a.c. spectrum can be larger than the positive half-line as can be easily seen upon taking $V = -1$ on half-space and $V = 0$ on the complement.

We need to start with some preliminary results. They will be mostly concerned with the study of the kernel of the operator $P_R^{-1}(k) = (-\Delta + k\lambda V_R + k^2)^{-1}$ where $V_R(x) = V(x) \cdot \omega_R(|x|)$. The existence of $P^{-1}(k)$ for $\text{Im } k > 0$ as a bounded operator from $L^2(\mathbb{R}^3)$ to $L^2(\mathbb{R}^3)$ was proved in [3]. Denote the kernel of $P_R^{-1}(x, y, k)$ by $K_R(x, y, k)$ and compare it to the free Green's kernel $L^0(x, y, k)$ in the following way. For $k = i$, we introduce the amplitude

$$b_R(\theta) = \lim_{r \rightarrow \infty} \frac{K_R(0, r\theta, i)}{L^0(0, r, i)}$$

where $\theta \in \mathbb{S}^2$. If $L_R(x, y, k)$ is the Green's function for $(-\Delta + |V_R| - k^2)^{-1}$, then similarly

$$a_R(\theta) = \lim_{r \rightarrow \infty} \frac{L_R(0, r\theta, i)}{L^0(0, r, i)}, \quad a(\theta) = \lim_{R \rightarrow \infty} a_R(\theta)$$

In [2], the following formulas were proved

$$(1.4) \quad \int_{\mathbb{S}^2} a(\theta) d\theta = C_1 \mathbb{E}_{X_0=0} \left[\exp \left(-\frac{1}{2} \int_0^\infty |V(X_\tau)| d\tau \right) \right]$$

For $\theta \in \mathbb{S}^2$, let

$$(1.5) \quad dG_t = \theta dt + dB_t$$

Then

$$(1.6) \quad a(\theta) = C_2 \mathbb{E}_{G_0=0} \left[\exp \left(-\frac{1}{2} \int_0^\infty |V(G_\tau)| d\tau \right) \right]$$

The condition (1.3) yields

$$\int_{\mathbb{S}^2} a(\theta) d\theta > 0$$

and thus $a(\theta) > 0$ for $\theta \in \Omega \subseteq \mathbb{S}^2$ and $|\Omega| > 0$. In particular, that means

$$(1.7) \quad \mathbb{E}_{G_0=0} \left[\exp \left(-\frac{1}{2} \int_0^\infty |V(G_t)| dt \right) \right] > 0$$

for any $\theta \in \Omega$. The bound (1.7) is exactly what we are going to use in this paper.

The first step is to prove an analog of the formula (1.6) for the function $b_R(\theta)$. The lemma below holds for any λ so we take $\lambda = 1$ for the shorthand.

Lemma 1.1. *If G_t is defined by (1.5), then*

$$(1.8) \quad b_R(\theta) = C_2 \mathbb{E}_{G_0=0} \left[\exp \left(-\frac{i}{2} \int_0^\infty V_R(G_\tau) d\tau \right) \right]$$

for any $R > 0$.

Proof. We need the following

Proposition 1.1. *Let Σ_r and B_r denote the sphere and closed ball of radius r both centered at the origin. If V is continuous and real-valued in B_r and $F(x) \in C(B_r)$, then $\phi(x)$, the solution to*

$$\frac{1}{2}\Delta\phi + \phi_{x_1} - \frac{i}{2}V\phi = -F, \quad \phi|_{\Sigma_r} = 0$$

admits the following representation

$$(1.9) \quad \phi(x) = \mathbb{E}_{G_0=x} \left[\int_0^T \exp \left(-\frac{i}{2} \int_0^t V(X(G_\tau)) d\tau \right) F(G_t) dt \right]$$

where $G_t = t(1, 0, 0) + B_t$, $G_0 = x$ and T is the exit time.

Notice that the solution ϕ always exists as the boundary problem can be reduced to inverting the operator $-\Delta + 1 + iV$ with Dirichlet boundary condition. This invertibility is a simple corollary of the spectral theory for hyperbolic pencils and it was proved in [3] in the context of the operators on the whole space. The existence of the expectation in the right hand side of (1.9) is guaranteed by $V \in \mathbb{R}$ and the fact that all trajectories $\{G_t\}$ are continuous almost surely and the exit time distribution has a small tail.

Proof. (proposition 1.1) This proof is quite standard for negative potentials (see [7], p.145 and [11], lemma 7.3.2) but we present it here for the reader's convenience. Take any $x, |x| < r$ and consider

$$\Xi_t = \phi(G_t) \exp \left(-\frac{i}{2} \int_0^t V(G_\tau) d\tau \right), \quad G_0 = x, \quad t < T$$

By Itô's calculus,

$$d\Xi_t = \exp \left(-\frac{i}{2} \int_0^t V(G_\tau) d\tau \right) d\phi(G_t) - \frac{i}{2} \phi(G_t) V(G_t) \exp \left(-\frac{i}{2} \int_0^t V(G_\tau) d\tau \right) dt$$

as

$$d\phi(G_t) = \left(\phi_{x_1}(G_t) + \frac{\Delta}{2} \phi(G_t) \right) dt + \phi_{x_1}(G_t) dB_t$$

and

$$d \exp \left(-\frac{i}{2} \int_0^t V(G_\tau) d\tau \right) = -\frac{i}{2} V(G_t) \exp \left(-\frac{i}{2} \int_0^t V(G_\tau) d\tau \right) dt$$

Since $\Xi_0 = \phi(0)$, we have

$$\begin{aligned} \mathbb{E}_x(\Xi_T) &= \phi(x) + \mathbb{E}_x \left[\int_0^T -F(G_t) \exp \left(-\frac{i}{2} \int_0^t V(G_\tau) d\tau \right) dt \right] \\ &\quad + \mathbb{E}_x \left[\int_0^T \phi_{x_1}(G_t) \exp \left(-\frac{i}{2} \int_0^t V(G_\tau) d\tau \right) dB_t \right] \end{aligned}$$

and the last term is equal to zero. As the left hand side is equal to zero as well due to the Dirichlet boundary conditions imposed, we have the statement of the proposition. \square

Now the proof of the lemma repeats the proof of the formula (2.8) (theorem 2.1, [2]) word for word. \square

Assume that $\theta \in \Omega$ so (1.7) holds. Consider truncations $V^{(\rho)}(x) = V(x) \cdot (1 - \omega_\rho(|x|))$.

Lemma 1.2. *If $\theta \in \Omega$, then*

$$(1.10) \quad \mathbb{E}_{G_0=0} \left[\exp \left(-\frac{1}{2} \int_0^\infty |V^{(\rho)}(G_t)| dt \right) \right] \rightarrow 1$$

as $\rho \rightarrow \infty$.

Proof. Take any $0 < R_1 < R_2$ and introduce t_1 , the random time of hitting the sphere $|x| = R_1$ for the first time. Then, denoting by $\tilde{G}(t)$ the solution to (1.5) with initial condition G_{t_1} , we have elementary inequality

$$(1.11) \quad \begin{aligned} \mathbb{E}_{G_0=0} \left[\exp \left(-\frac{1}{2} \int_0^{t_1} |V_{R_1/2}(G_t)| dt \right) \mathbb{E}_{G_{t_1}} \left[\exp \left(-\frac{1}{2} \int_0^\infty |V^{(R_2)}(\tilde{G}_t)| dt \right) \right] \right] \\ \geq \mathbb{E}_{G_0=0} \left[\exp \left(-\frac{1}{2} \int_0^\infty |V(G_t)| dt \right) \right] \end{aligned}$$

The trajectory G_t is a linear drift plus 3d Browning motion oscillation thus for fixed R_1 we have decoupling

$$(1.12) \quad \begin{aligned} \lim_{R_2 \rightarrow \infty} \left(\mathbb{E}_{G_0=0} \left[\exp \left(-\frac{1}{2} \int_0^{t_1} |V_{R_1/2}(G_t)| dt \right) \mathbb{E}_{G_{t_1}} \left[\exp \left(-\frac{1}{2} \int_0^\infty |V^{(R_2)}(\tilde{G}_t)| dt \right) \right] \right] \right) \\ = \mathbb{E}_{G_0=0} \left[\exp \left(-\frac{1}{2} \int_0^{t_1} |V_{R_1/2}(G_t)| dt \right) \right] \cdot \gamma \end{aligned}$$

with

$$\gamma = \lim_{R_2 \rightarrow \infty} \left(\mathbb{E}_{G_0=0} \left[\exp \left(-\frac{1}{2} \int_0^\infty |V^{(R_2)}(G_t)| dt \right) \right] \right)$$

On the other hand,

$$\mathbb{E}_{G_0=0} \left[\exp \left(-\frac{1}{2} \int_0^{t_1} |V_{R_1/2}(G_t)| dt \right) \right] \rightarrow \mathbb{E}_{G_0=0} \left[\exp \left(-\frac{1}{2} \int_0^\infty |V(G_t)| dt \right) \right], \quad R_1 \rightarrow \infty$$

which along with (1.11) and (1.12) implies $\gamma = 1$. \square

Remark. Let $a^{(\rho)}$ denote an amplitude for the potential $V^{(\rho)}$. The lemma then says that $a^{(\rho)}(\theta) \rightarrow 1$ as $\rho \rightarrow \infty$ for any $\theta \in \Omega$. Notice also that the lemma is wrong in general if the trajectory G_t is replaced by X_t as can be easily seen by letting $V = 1$ on the half-space and $V = 0$ on the complement.

Now we are ready for the proof of theorem 1.2.

Proof. (theorem 1.2)

Notice first that the standard trace-class perturbation argument [12] implies that $\sigma_{ac}(-\Delta + V_\lambda) = \sigma_{ac}(-\Delta + V_\lambda^{(\rho)})$ for any ρ . Fix some large $c > 0$ and take $\lambda \in [-c, c]$. Then, by lemma 1.2, we can make ρ large enough so that for any $\theta \in \Omega_1 \subseteq \Omega$, we have

$$\mathbb{E} \left[\exp \left(-\frac{c}{2} \int_0^\infty |V^{(\rho)}(G_t)| dt \right) \right] > 0.99$$

Then, due to (1.8),

$$(1.13) \quad |b_R^{(\rho)}(\theta)| > 1/2$$

for any $\theta \in \Omega_1$ and any $R > \rho$. Now, we need to recall several results from [3] and repeat a couple of arguments from this paper. Let $f = \chi_{|x|<1}$. Denote the spectral measure of f with respect to $-\Delta + \lambda V_R^{(\rho)}$ by $\sigma_{\rho,R}(E, \lambda)$. For $k \in \mathbb{C}^+$, consider

$$J_{\rho,R}(k, \theta, \lambda) = \lim_{r \rightarrow \infty} \frac{\left((-\Delta + \lambda k V_R^{(\rho)} + k^2)^{-1} f \right)(r\theta)}{r^{-1} e^{ikr}}$$

Then, the formula (38) from [3] says

$$(1.14) \quad \sigma'_{\rho,R}(k^2, k\lambda) = k\pi^{-1} \|J_{\rho,R}(k, \theta, \lambda)\|_{L^2(\mathbb{S}^2)}^2$$

where $k \neq 0$ is real and in the right hand side the limiting value of $B_{\rho,R}$ as $\text{Im } k \rightarrow +0$ is taken. This function $J_{\rho,R}(k, \theta, \lambda)$ is continuous on $\overline{\mathbb{C}^+} \setminus \{0\}$ as seen from the absorption principle ([13], chapter 13, section 8 or [3]). Around zero, we have an estimate

$$(1.15) \quad |J_{\rho,R}(k, \theta, \lambda)| < C(\rho, R) |k|^{-1}$$

that can be deduced from the representation

$$P^{-1}(k)f = R_0(k^2)f - k\lambda R_0(k^2)V_R^{(\rho)}P^{-1}(k)f \quad k \in \mathbb{C}^+$$

and an estimate

$$\|P^{-1}(k)\| \leq (\text{Im } k)^{-2}$$

(see (37), [3]).

For large $|k|$, we have the following uniform estimate

$$(1.16) \quad \int_{\mathbb{S}^2} |J_{\rho,R}(k, \theta, \lambda)|^2 d\theta < C \frac{1 + |k| \text{Im } k}{[\text{Im } k]^4} \|f(x)\|_2 \|f(x)e^{2\text{Im } k|x|}\|_2$$

(take $r \rightarrow \infty$ in (48), [3]).

Now, consider the function

$$g(k) = \ln \|ke^{2ik} J_{\rho,R}(k, \theta, \lambda)\|_{L^2(\mathbb{S}^2)}$$

This function is subharmonic in \mathbb{C}^+ and the estimates (1.15) and (1.16) enable us to apply the mean-value inequality with the reference point $k = i$.

This, along with the identity (1.14), gives

$$\int_{\mathbb{R}} \frac{\ln \sigma'_{\rho,R}(k^2, k\lambda)}{k^2 + 1} dk > C_1 + C_2 \ln \int_{\mathbb{S}^2} |J_{\rho,R}(i, \theta, \lambda)|^2 d\theta, \quad C_2 > 0$$

Now, notice that the choice of f guarantees that

$$|J_{\rho,R}(i, \theta, \lambda)| \sim |b_R^{(\rho)}(i, \theta, \lambda)|$$

and (1.13) implies that

$$C_1 + C_2 \ln \int_{\mathbb{S}^2} |J_{\rho,R}(i, \theta, \lambda)|^2 d\theta > C_3$$

uniformly in R and $\lambda \in [-c, c]$. Thus, we have an estimate

$$\int_{\mathbb{R}} \frac{\ln \sigma'_{\rho,R}(k^2, k\lambda)}{k^2 + 1} dk > -C$$

uniformly in $R > \rho$ and $\lambda \in [-c, c]$ with any fixed c . Now, taking any interval $(a, b) \subset (0, \infty)$, we have

$$\int_a^b dE \int_{-c}^c \ln \sigma'_{\rho, R}(E, \lambda) d\lambda > -C$$

uniformly in R so taking $R \rightarrow \infty$ and using the lower semicontinuity of the entropy (see [8] and [3], p.21 and Lemma 3.4), we get

$$\int_a^b dE \int_{-c}^c \ln \sigma'_\rho(E, \lambda) d\lambda > -C$$

The Fubini-Tonelli theorem now gives

$$\int_a^b \ln \sigma'_\rho(E, \lambda) dE > -\infty$$

for a.e. λ so $[a, b] \subseteq \sigma_{ac}(H_\lambda^{(\rho)})$ for a.e. λ . As was mentioned already, the a.c. spectrum is stable under changing the potential on any compact set. Thus, $[a, b] \subseteq \sigma_{ac}(H_\lambda)$ for a.e. λ . Since $[a, b]$ was taking arbitrarily, we have statement of the theorem. \square

Remark. In the paper [2], we studied the case when the potential $V \geq 0$ and is supported on the set E (a good example to think about is a countable collection of balls) of any geometric structure. The special modified capacity and the harmonic measure were introduced and studied which allowed the effective estimation of probabilities in the natural geometric terms. The same results, e.g. the estimates in terms of anisotropic Hausdorff content and the size of the spherical projection, are true in the current setting when the potential is not assumed to be positive. The statements however are true only generically in λ .

Acknowledgement. We acknowledge the support by Alfred P. Sloan Research Fellowship and the NSF grant DMS-0758239. Thanks go to Stas Kupin and the University of Provence where part of this work done.

REFERENCES

- [1] M. Abramowitz, I. Stegun, Handbook of mathematical functions with formulae, graphs, and mathematical tables. National Bureau of Standards, Applied Mathematics Series, 55, 1964 (tenth printing, December 1972, with corrections).
- [2] S. Denisov, S. Kupin, Itô diffusions, modified capacity, and harmonic measure. Applications to Schrödinger operators, to appear in IMRN.
- [3] S. Denisov, Schrödinger operators and associated hyperbolic pencils, J. Funct. Anal., Vol. 254, 2008, 2186-2226.
- [4] S. Denisov, A. Kiselev, Spectral properties of Schrödinger operators with decaying potentials. B. Simon Festschrift, Proceedings of Symposia in Pure Mathematics, vol. 76.2, AMS 2007, pp. 565-589.
- [5] S. Denisov, On the preservation of the absolutely continuous spectrum for Schrödinger operators. J. Funct. Anal. 231 (2006), 143-156.
- [6] S. Denisov, Absolutely continuous spectrum of multidimensional Schrödinger operator. Int. Math. Res. Not. 74 (2004), 3963-3982.
- [7] A. Friedman, Stochastic differential equations and applications. Vol. 1. Probability and Mathematical Statistics, Vol. 28. Academic Press, 1975.
- [8] R. Killip, B. Simon, Sum rules for Jacobi matrices and their applications to spectral theory. Ann. of Math. (2) 158 (2003), no. 1, 253-321.

- [9] A. Laptev, S. Naboko, O. Safronov, A Szegő condition for a multidimensional Schrödinger operator. J. Funct. Anal. 219 (2005), no. 2, 285-305.
- [10] A. Laptev, S. Naboko, O. Safronov, Absolutely continuous spectrum of Schrödinger operators with slowly decaying and oscillating potentials. Comm. Math. Phys. 253 (2005), no.3, 611-631.
- [11] B. Øksendal, Stochastic differential equations. Springer-Verlag, 2003 (sixth edition).
- [12] M. Reed, B. Simon, "Methods of modern mathematical physics. Vol. 3, Scattering theory", Academic Press, New York-London, 1979.
- [13] M. Reed, B. Simon, Methods of modern mathematical physics. IV. Analysis of operators. Academic Press, New York-London, 1978.
- [14] O. Safronov, Absolutely continuous spectrum of multi-dimensional Schrödinger operators with slowly decaying potentials. Spectral theory of differential operators, 205214, Amer. Math. Soc. Transl. Ser. 2, 225, Amer. Math. Soc., Providence, RI, 2008.
- [15] O. Safronov, Absolutely continuous spectrum of a one-parametric family of Schrödinger operators, preprint, 2011.
- [16] D. Yafaev, Scattering theory: some old and new problems, Lecture Notes in Mathematics, 1735. Springer-Verlag, Berlin, 2000.

MATHEMATICS DEPARTMENT, UNIVERSITY OF WISCONSIN-MADISON, 480 LINCOLN DR., MADISON, WI 53706 USA
E-mail address: `denissov@math.wisc.edu`